Journal of Novel Applied Sciences
Available online at www.jnasci.org
©2014 JNAS Journal-2014-3-11/1303-1307
ISSN 2322-5149 ©2014 JNAS

Product of pairwise permutable nilpotent n subgroups $A_1,\ldots,A_t$.

Behnam Razzaghmaneshi

Asistant Professor of Department of Mathematics and Computer science, Islamic Azad University Talesh Branch, Talesh, Iran

***Corresponding author*:** Behnam Razzaghmaneshi

ABSTRACT: Let the finite group $G=A_1\ldots A_t$ be the product of pairwise permutable nilpotent subgroups $A_1,\ldots,A_t$. Then G is soluble.

***Keywords*:** finite group, product of pairwise permutable nilpotent subgroups, soluble group.

INTRODUCTION

FIn 1940 G. Zappa(see (24)) and in 1950 J.Szip (see (23)) studied bout products of groups concerned finite groups. In 1961 O.H.Kegel (See (8)) and in 1958 H.Wielandt (see (10)) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups .

In 1955 N.Itô (see (7)) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. P.M. Cohn (1956) (see(21)) and L.Redei (1950)(see (22)) considered products of cyclic groups, and around 1965 O.H.Kegel (See (30) & (31)) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (see (19)) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972 (See (20)&(1)). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his Habilitationsschrift (1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition $\mathfrak{X}$, when does G have the same finiteness condition $\mathfrak{X}$?(See (20))

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see (1), (2),(3),(4) and (6)) , N.S. Chernikov (see (5)), S. Franciosi, F. de Giovanni (see (3),(6),(32),(33),(34),(35), and (36)), O.H.Kegel (see (8)), J.C.Lennox (see (12)) , D.J.S. Robinson(see (9) and (15)), J.E. Roseblade(see (13)), Y.P.Sysak(see (37),(38),(39)and(40)), J.S.Wilson (see (41)), and D.I.Zaitsev(see (11) and (18)).

Now, in this paper, we study the residual finite group and min-by-max subgroups of the group G and its relations, and the end we prove that if $A_1,A_2,\ldots,A_n$, are finitely many pairwise permutable abelian min-by-max subgroups of the group G such that G is the products of $A_1,\ldots,A_n$. Then G is soluble min-by-max-group and J(G) is products of $J(A_1),\ldots,J(A_n)$, i.e. $J(G)=J(A_1)\ldots J(A_n)$. For do this, in chapter 2 we express the elementary lemmas and theorems and in chapter three we prove the main theorem. In this chapter we express the elementary Lemma and Diffinitons whose used the prove the mantheorem in chapter 3.

2. Priliminaries : (elementary properties and theorems.)

In this section we study the elementary Lemma and theorems, whose using in section 3 and prove of main theorem.

2.1. Lemma:

(See 25) Let the groups $G=\bigcup_{i\in I}^{t} H_i x_i,$ where $H_1,\ldots,H_t$ are subgroups of G. Then at least one of the subgroup H_i has finite index in G.

Proof :

Let s be the number of distinct subgroups among $H_1,\ldots,H_t$. If s=1, the lemma is clear. Suppose that s>1, and let I be the set of indices i such that $H_i=H_1.$ If $G=\bigcup_{i\in I} H_1 x_i,$ then H_i has finite index in G. Assume now that there is an element y in $G\setminus\bigcup_{i\in I} H_1 x_i.$ Thus the intersection $H_1 y\cap(\bigcup_{i\in I} H_1 x_i)$ is empty, and hence $H_1 y=\bigcup_{j\notin I} H_j x_j.$ Therefore for each i in I we obtain $H_1 x_i\subseteq\bigcup_{j\notin I} H_j x_j y^{-1} x_i.$ This proves that G is the union of finitely many cosets of the subgroups H_j, where j is not in I. As the number of distinct subgroups among these is s-1, by induction on s at least one of them has finite index in G.

2.2. Lemma:

Let the group G=AB be the product of two subgroups A and B. If A_0 and B_0 are subgroups of finite index of A and B, respectively, then the subgroup H=<A_0, B_0> has index at most mn in G, where $|A:A_0|=m$ and $|B:B_0|=n$.

Proof :

Let $\{a_1,\ldots,a_m\}$ be a left transversal of A_0 in A and $\{b_1,\ldots,b_n\}$ a right transversal of B_0 in B. Then.

G=AB= $\bigcup_{i,j} a_i A_0 B_0 b_i=\bigcup_{i,j}(a_i H a_i^{-1})a_i b_i$

is the union of finitely many right cosets of conjugates of H. It follows from Lemma 2.1 that H has finite H has finite index in G. To obtain the required bound for |G:H|, it is clearly enough to consider the finite factor group G/H_G, where H_G is the core of H in G. Consequently we may suppose that G is finite. Then.

$$|G|=\frac{|A|.|B|}{|A\cap B|}\le\frac{|A|.|B|}{|A_0\cap B_0|}=\frac{|A_0|.|B_0|}{|A_0\cap B_0|}mn\le|H|\,mn,$$

And so $|G:H|\le mn.$

2.3. Lemma:

(See (1)) Let the group G=AB be the product of two subgroups A and B.

(i) If A and B satisfy the maximal condition on subgroups, then G satisfies the maximal condition on normal subgroups.

(ii) If A and B satisfy the minimal condition on subgroups, then G satisfies the minimal condition on normal subgroups.

Proof:

(i) Let $(H_n)_{n\in N}$ be an ascending sequence of normal subgroups of G. Then $(A\cap H_n)_{n\in N}$ and $(B\cap H_n)_{n\in N}$ are ascending sequences of subgroups of A and B, respectively. Hence

$$A\cap H_n = A\cap H_{n+1} \quad and \quad B\cap AH_n = B\cap AH_{n+1}$$

For almost all n. It follows that

$$AH_n = AB\cap AH_n = A(B\cap AH_n) = A(B\cap AH_{n+1}) = AH_{n+1}$$

And so

$$H_n = H_n(A\cap H_{n+1}) = AH_n\cap H_{n+1} = AH_{n+1}\cap H_{n+1} = H_{n+1}$$

For almost all n. Therefore G satisfies the maximal condition on normal subgroups.
The proof of (ii) is similar.

2.4.Lemma:

Let the group G=AB be the product of two subgroups A and B. If x, y are elements of G, then G=A^xB^y. Moreover, there exists an element z of G such that $A^x=A^z$ and $B^y=B^z$.

Proof :

Write xy^{-1}=ab with a in A and b in B. If z=a^{-1}x, then x=az and $y=b^{-1}z,$ so that $A^x=A^z$ and $B^y=B^z$. It follows that G= A^z B^z= A^x B^y.

2.5. Difinition :

Recall that a finite group is a D_π - groups if evety π-subgroup is contained in a Hall π-subgroup and any two Hall π-subgroups are conjugate.

2.6. Lemma:

Let the finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π - group, for a set π of primes, then there exist Hall π-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π-subgroups of G.

Proof:

Let A_1, B_1, and G_1 be Hall π-subgroups of A, B, and G, respectively. Since G is a D_π - group, there exist elements x and y such that A_1^x and B_1^y are both contained in G_1. It follows from Lemma 2.4 that $A^x=A^z$ and $B^y=B^z$ for some z in G. Thus $A_0=A_1^{xz^{-1}}$ and $B_0=B_1^{yz^{-1}}$ are Hall π-subgroups of A and B, respectively, which are both contained in $G_0=G_1^{z^{-1}}$. Clearly the order of $A_0\cap B_0$ is bounded by the maximum π-divisor n of the order of $A\cap B$ since $|G|=\frac{|A|.|B|}{|A\cap B|}$, It follows that $|G_0|=\frac{|A_0|.|B_0|}{n}\leq\frac{|A_0|.|B_0|}{|A_0\cap B_0|}=|A_0B_0|$. Therefore $A_0B_0=G_0$ is a Hall π-subgroup of G.

2.7. Corollary:

Let the finite group G=AB be the product of two subgroups A and B .Then for each prime p there exist Sylow p-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Sylow p-subgroup of G.

Proof:

See (5)

2. 8. Corollary **:**

Let the finite group G=AB=AK=BK be the product of three nilpotent subgroups, A,B, and K, where K is normal in G. Then G is nilpotent .

Proof:

See((4), corollary 1.3.5)

2.9. Theorem *:*

(See (7)) Let the group G=AB be the product of two abelian subgroups A and B. Then G is metabelian.

Proof :

Let a,a_1 be elements of A and b, b_1 elements of B. Write $b^{a_1}=a_2b_2$ *and* $a^{b_1}=b_3a_3$, *whit* a_2, a_3 *in* A *and* b_2,b_3 *in* B. Then

$$[a,b]^{b_1a_1}=[a,b^{a_1}]^{b_1}=[a,b_2]^{b_1}=[a^{b_1},b_2]=[a_3,b_2]$$

and

$$[a,b]^{b_1a_1}=[a^{b_1},b]^{a_1}=[a_3,b]^{a_1}=[a_3,b^{a_1}]=[a_3,b_2].$$

This proves that the commutators (a,b) and (a_1,b_1) commute. Since the factor group G/(A,B) is abelian, it follows that $G'=[a,b],$ and hence G' is abelian.

2.10 .Lemma:

Let the group G=AB be the product of two abelian subgroups A and B, and let S be a factorized subgroup of G. Then the centralizer $C_G(S)$ is factorized . Moreover, every term of the upper central series of G is factorized.

Proof:

Since S is factorized, we have that S=$(A\cap S)(B\cap S).$ Let x=ab be an element of S, where a is in $A\cap S$ and b is in $B\cap S$ .If $c=a_1b_1$ is an element of $C_G(S)$, with a_1 in A and b_1 in B, it follows that.

$$[a_1,x]=[a_1,ab]=[a_1,b]=[cb_1^{-1},b]=[c,b]^{b_1^{-1}}=1.$$

Therefore a_1 belongs to $C_G(S)$, and $C_G(S)$ is factorized by Lemma 1.1.1 of (4). In particular, the center of G is factorized. It follows from Lemma 1.1.2 of (4) that also every term of the upper central series of G is factorized.

2.11. Lemma:

Let the group G=AB be the product of two subgroups A and B. If A_1, B_1, and F are the FC-centers of A, B, and C, respectively, then $F=A_1F\cap B_1F$. In particular, if A and B are FC-groups, the FC-centre of G is factorized subgroup.

Proof :

Let x be an element of $A_1F\cap B_1F$, and write x=au where a is in A_1 and u is in F. Since the centralizers $C_A(a)$ and $C_A(u)$ have finite index in A, the index $|A: C_A(x)|$ is also finite. Similarly, $C_B(x)$ has finite index in B. Therefore $|G:\langle C_A(x),C_B(x)\rangle|$ is finite by Lemma 1.2.5 of (4). It follows that $C_G(x)$ has finite index in G and hence x belongs to F. Thus $F=A_1F\cap B_1F$.

2.12. Lemma:

(See (7)) Let the finite non-trivial group G=AB be the product of two abelian subgroups A and B. Then there exists a non-trivial normal subgroup of G contained in A or B.

Proof :

Assume that {1} is the only normal subgroup of G contained in A or B. By Lemma 2.11 have Z(G)=(A $\cap Z(G))(B\cap Z(G))=1.$ The centralizer $C=C_G(A\cap C_G(G'))$ *contains* $AG',$ and so is normal in G.

Since $B\bigcap(AZ(C))\leq Z(G)=1,$ it follows that $AZ(C)=A(B\bigcap AZ(C))=A.$ This Z(G) is a normal subgroup of G contained in A, and so Z(G)=1. Since G' is abelian by Theorem 2.9, we have $A\bigcap G'\leq A\bigcap C_G(G')\leq Z(C)=1.$

Similarly $B\bigcap G'\leq B\bigcap C_G(G')\leq Z(C)=1.$ The factorizer $X=X(G')$ has the triple factorization $X=A^*B^*=A^*G'=B^*G'$, Where $A^*=A\bigcap BG'$ *and* $B^*=B\bigcap AG'.$ Thus X is nilpotent by Corollary 2.8, so that

$$Z(X)=(A\bigcap Z(X))(B\bigcap Z(X))$$

is not trivial. Hence there exists a non-trivial normal subgroup N of X contained in A or B. Suppose that N is contained in A. Since G' normalizes N, we have $[N,G']\leq N\bigcap G'\leq A\bigcap G'=1.$ Therefore we obtain the contradiction $N\leq A\bigcap G_G(G')=1.$

3. Main Theorem:

in this section by using of sections 1 and 2, we prove the following main theorem.

3.1. Theorem:

Let the finite group $G=A_1\ldots A_t$ be the product of pairwise permutable nilpotent subgroups $A_1,\ldots,A_t$. Then G is soluble.

Proof:

Let p be a prime, and for every i=1…,t let P_1 be the unique Sylow p-complement of A_i. If $i\neq j$, the subgroup A_iA_j is soluble by Theorem 2.4.3 of (4). Hence it follows from Lemma 2.6,that P_iP_j is a Sylow p-complement of A_iA_j. Thuse the subgroups $P_1,\ldots,P_t$ pairwise permute, and the product $P_1P_2\ldots P_t$ is a Sylow p-complement of G. Since G has a Sylow p-complement for every prime p, it is soluble.

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